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THE ASYMPTOTIC EXPANSION OF INTEGRAL FUNCTIONS DEFINED BY TAYLOR SERIES

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1·1. Considerable attention has been devoted to the behaviour of the general integral function for large values of the variable, and many important theorems have been proved in this field. On the other hand, the behaviour of a large number of particular integral functions has been studied in detail and their asymptotic expansions for certain regions of the plane obtained. There is, however, a substantial gap between the two theories. For example, much of the most interesting work on the general integral function deals with the distribution of its zeroes and other values; but many of the asymptotic expansions obtained for particular functions are not valid in the regions in which these functions have zeroes.

In this paper and its sequels I propose to study several fairly wide classes of functions defined by Taylor series; from the properties of the coefficients I deduce asymptotic expansions of the function defined by the series. For the sort of functions I consider we can usually divide the whole complex plane, with the exception of certain "barrier regions", into a number of regions R, in each of which the function is given asymptotically by an equation of the form

$$f(x) = \psi_R(x)(1+\rho_x), \tag{1·11}$$

where $\psi_R(x)$ is an elementary function of x, such as an exponential or a power, with no zeroes in R, and ρ_x tends uniformly to zero as $|x|$ tends to infinity in R. It then follows that $f(x)$ has only a finite number of zeroes in the region R and that the zeroes are to be looked for in the barrier region between two regions R and R', where we may hope to prove that

$$f(x) = \psi_R(x)\,(1+\rho_x)+\psi_{R'}(x)\,(1+\rho'_x). \tag{1·12}$$

It is usually more difficult to prove a result of this type than one of the type (1·11), but, if we can do so, we can then determine the asymptotic distribution of the zeroes and other values of $f(x)$ very precisely. The deduction of information about this distribution from the properties of the coefficients of the Taylor series is of course one of the central problems of the theory of integral functions.

For such a purpose and for other applications it is necessary to prove that each asymptotic expansion is valid uniformly in the appropriate region and to ensure that these regions completely cover the plane. The point is best illustrated by an example. For a particular function the expansion valid for $|\arg x| < \frac{1}{2}\pi$ (but valid uniformly only for $|\arg x| \leqslant \frac{1}{2}\pi - \epsilon$ for every $\epsilon > 0$) and the expansion valid for $|\arg(-x)| \leqslant \frac{1}{2}\pi$ are already known; but for many applications these are insufficient. We are not entitled to use either expansion as we pass to infinity along the line* $\mathscr{R}(x) = 1$, nor can we determine the distribution of zeroes near the imaginary axis.

The determination of expansions valid in the barrier regions and the proof that each expansion is valid uniformly in the appropriate region add greatly to the length and complication of the work and thus obscure the fundamental simplicity of the method. But both these points are of great importance, and their careful investigation leads to many of the essentially new results found here.

1·2. The particular type of function with which I deal here is that defined by

$$f(x) = c_0 + c_1 x + c_2 x^2 + \dots,$$

where
$$c_n = \frac{\phi(n)}{\Gamma(\kappa n + \beta)},$$

κ and β may be real or complex, $\mathscr{R}(\kappa) > 0$, and $\phi(t)$ is a function satisfying suitable conditions.

Mittag-Leffler (1905), Wiman (1905), Barnes (1906, 1907), Hardy (1905), Fox (1928), Wright (1934) and other writers have found asymptotic expansions of functions which are particular cases of $f(x)$. Mittag-Leffler and Wiman discussed in detail the asymptotic behaviour of the function

$$E_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + 1)}.$$

Their results are included in those of my Lemma 6 for the more general function

$$\sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\kappa n + \tau)}.$$

Hardy obtained expansions of his function

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+a)^s \, n!}$$

for the whole plane, including the barrier regions; this was a preliminary to his main purpose, viz. the study of the zeroes. Barnes found expansions for various special examples of our function, but not those valid in the barrier regions. Fox's work on the generalized hypergeometric function applies to part (but not all) of the barrier regions; his method requires κ to be real and rational. All these results are special cases of those found here. In a previous paper (Wright 1935), I gave complete results (without proof) for the generalized hypergeometric function; these are in part corollaries of the theorems of this paper, and their proof will be completed elsewhere.

* We use $\mathscr{R}(y)$ to denote the real part of y.

The most general results concerning the asymptotic expansion of $f(x)$ so far obtained are due to Watson (1913), to whose work I am greatly indebted. He takes

$$c_n = \frac{\phi(n)\, e^{gn}}{\Gamma(\kappa n+\beta)},$$

but it is clear that there is no real loss of generality in taking $g=0$; otherwise we have only to replace x by $e^g x$ in our results. Watson imposes stricter conditions on $\phi(t)$ than I do here; apart from this, his results correspond roughly to my Theorems 2, 4 and 6, and do not apply to the barrier regions referred to above. In view of the importance of his work, I make a detailed comparison of our results at a later stage (§ 2·5).

1·3. We suppose that $\dfrac{\phi(t)}{\Gamma(\kappa t+\beta)}$ is defined for $t=0, 1, 2, 3, \ldots$, though if, for example, $\dfrac{\phi(t)}{\Gamma(\kappa t+\beta)}$ has a pole at $t=n_1$, where n_1 is a non-negative integer, our proofs and results will be unchanged if $c_{n_1} x^{n_1}$ is replaced in $f(x)$ by the residue of

$$\frac{\pi}{\sin \pi t}\frac{\phi(t)\,(-x)^t}{\Gamma(\kappa t+\beta)}$$

at its multiple pole at $t=n_1$.

We write $\delta = |\kappa|$ and $\gamma = \arg \kappa$. Since $\mathscr{R}(\kappa)>0$, we may always take $|\gamma|<\frac{1}{2}\pi$. We choose $\arg x$ so that

$$-\pi<\arg x-\tan\gamma \log|x|\leqslant\pi, \tag{1·31}$$

and $\arg(-x)$ so that

$$-\pi<\arg(-x)-\tan\gamma \log|x|\leqslant\pi. \tag{1·32}$$

With this determination of $\arg x$ we write

$$X_0 = X = x^{1/\kappa}, \quad X_s = Xe^{2s\pi i/\kappa},$$

where s is any integer. It follows that

$$\delta \arg X = \cos\gamma \arg x - \sin\gamma \log|x|, \tag{1·33}$$

so that

$$-\frac{\pi\cos\gamma}{\delta}<\arg X\leqslant\frac{\pi\cos\gamma}{\delta}. \tag{1·34}$$

The numbers ϵ and ϵ', to be thought of as small, are any assigned positive numbers; the real numbers h_1, h_2, h are related to the properties of $\phi(t)$. We use K to denote a positive number, independent of x (and so of X), of the real variables r, θ, v and w and of the complex variables y, u and t, but possibly depending on $\kappa, \beta, \tau, \epsilon, \epsilon', \sigma, \sigma_1, \sigma_2, \mu, \mu', M, P, h_1, h_2, h$ and any other parameters. The value of K varies from one occurrence to another. We use K_1 and K_2 to denote fixed numbers (the same at each occurrence) of the type of K, and the notation $\psi = O(\chi)$ to denote that a number K exists such that $|\psi|<K|\chi|$. It is to be observed that K does not depend on $\arg x$ (or $\arg X$); in other words, statements involving K or $O(\ldots)$ are uniform in $\arg X$ throughout the interval of $\arg X$ in question.

By (1·31) we have $$\arg x = \tan\gamma \log|x| + O(1).$$

Hence $$\delta \log|X| = \cos\gamma \log|x| + \sin\gamma \arg x \tag{1·35}$$
$$= \sec\gamma \log|x| + O(1),$$

that is $$\mathscr{R}(\kappa) \log|X| = \log|x| + O(1).$$

It follows that $$x = O(X^{\mathscr{R}(\kappa)}), \quad X = O(x^{1/\{\mathscr{R}(\kappa)\}}). \tag{1·36}$$

1·4. If $\phi(t)$ satisfies suitable conditions, the form assumed by the asymptotic expansion of $f(x)$ depends on the value of $\arg X$. If $|\arg X| \leqslant \frac{1}{2}\pi - \epsilon$, the asymptotic expansion of $f(x)$ consists of one "exponentially large" expansion of the form

$$e^X\left\{\sum_{m=1}^{M} A_m X^{1-\alpha_m} + O(X^{1-\alpha_{M+1}})\right\}$$

or the sum of several such expansions with different X_s replacing X. In general

$$\log f(x) \sim X_s$$

for some s. If $|\arg X| \geqslant \frac{1}{2}\pi + \epsilon$, $f(x)$ has an "algebraic" asymptotic expansion and, in general,

$$f(x) \sim b(-x)^c,$$

where b, c are independent of x. On the other hand, if $\arg X$ is nearly $\pm\frac{1}{2}\pi$, the behaviour of $f(x)$ is less simple and the asymptotic expansion is of the "mixed" type (1·12), i.e. it consists of the sum of the two asymptotic expansions which are separately valid in the two neighbouring regions.

However, all the above values of $\arg X$ may not be possible. By (1·34),

$$|\arg X| \leqslant \frac{\pi\cos\gamma}{\delta} = \pi\mathscr{R}\left(\frac{1}{\kappa}\right).$$

There are now three cases.

(i) If $\mathscr{R}\left(\frac{1}{\kappa}\right) < \frac{1}{2}$ (i.e. if $|\kappa - 1| > 1$), we have

$$|\arg X| \leqslant \pi\mathscr{R}\left(\frac{1}{\kappa}\right) = \tfrac{1}{2}\pi - \pi\left\{\tfrac{1}{2} - \mathscr{R}\left(\frac{1}{\kappa}\right)\right\} = \tfrac{1}{2}\pi - \epsilon$$

for a positive ϵ. The asymptotic expansion of $f(x)$ consists of one exponentially large expansion or the sum of several such expansions. Theorem 2 gives complete information in this case.

(ii) If $\mathscr{R}\left(\frac{1}{\kappa}\right) = \frac{1}{2}$ (i.e. if $|\kappa - 1| = 1$), we have $|\arg X| \leqslant \frac{1}{2}\pi$. In the x-plane, the spirals $\arg X = \pm\frac{1}{2}\pi$, i.e.

$$\arg x - \tan\gamma \log|x| = \pm\tfrac{1}{2}\delta\pi \sec\gamma = \pm\pi, \tag{1·41}$$

coincide. Apart from the neighbourhood of the spiral (more precisely, throughout the region in which $|\arg X| \leqslant \frac{1}{2}\pi - \epsilon$), there is a single exponential expansion given by Theorem 4. Near the spiral there is a mixed expansion given by Theorem 8.

(iii) If $\mathscr{R}\left(\frac{1}{\kappa}\right) > \frac{1}{2}$ (i.e. if $|\kappa - 1| < 1$), we require three theorems. In the x-plane, the spirals $\arg X = \pm\frac{1}{2}\pi$, i.e.

$$\arg x - \tan\gamma \log|x| = \pm\tfrac{1}{2}\delta\pi \sec\gamma, \tag{1·42}$$

are distinct (since $\delta \sec\gamma < 2$) and divide the plane into two connected regions. In the interior of one of these regions (more precisely, when $|\arg X| \leqslant \frac{1}{2}\pi - \epsilon$) we have a single exponential expansion given by Theorem 4; in the interior of the other region, when $|\arg X| \geqslant \frac{1}{2}\pi + \epsilon$, the (algebraic) expansion is given by Theorem 6, while Theorem 7 gives the mixed expansion valid in the neighbourhood of the spirals.

When κ is real, the results are slightly simplified. In case (i) the expansion consists of just one exponential expansion, except near the negative half of the real axis, where the expansion is the sum of two exponential expansions. In case (ii), the spiral $\arg X = \frac{1}{2}\pi$ becomes the negative half of the real axis, while in case (iii) the spirals $\arg X = \pm\frac{1}{2}\pi$ become the straight lines $\arg x = \frac{1}{2}\pi\kappa$.

Formal statement of results

2·1. After this preliminary sketch we proceed to state our results precisely in the form of theorems. We shall say that $\phi(t)$ *satisfies condition A* for a certain set of values of t if an integer $M \geqslant 0$ and numbers

$$A_1, A_2, \ldots, A_M, \alpha_1, \alpha_2, \ldots, \alpha_{M+1}$$

exist such that $$\mathscr{R}(\alpha_1) \geqslant \mathscr{R}(\alpha_2) \geqslant \ldots \geqslant \mathscr{R}(\alpha_M) > \mathscr{R}(\alpha_{M+1})$$

and $$\frac{\phi(t)}{\Gamma(\kappa t+\beta)} = \sum_{m=1}^{M} \frac{\kappa A_m}{\Gamma(\kappa t+\alpha_m)} + O\left(\frac{1}{\Gamma(\kappa t+\alpha_{M+1})}\right) \tag{2·11}$$

for this set of values of t; if $M = 0$, (2·11) takes the form

$$\frac{\phi(t)}{\Gamma(\kappa t+\beta)} = O\left(\frac{1}{\Gamma(\kappa t+\alpha_1)}\right).$$

It is to be noted that we only require (2·11) to be true for one value of M.*

We write $$E(y) = \{\mathscr{R}(y)\}^{-\frac{1}{2}} \quad (\mathscr{R}(y) \geqslant 1), \quad E(y) = 1 \quad (\mathscr{R}(y) < 1),$$

$I(y)$ for the asymptotic expansion

$$ye^y\left\{\sum_{m=1}^{M} A_m y^{-\alpha_m} + O(y^{\frac{1}{2}-\alpha_{M+1}} E(y))\right\},$$

$H(-x)$ for the sum of the residues of

$$k(t) = -\frac{\pi\phi(t)\,(-x)^t}{\sin\pi t\,\Gamma(\kappa t+\beta)} \tag{2·12}$$

* Theorem 9 shows that $\phi(t)$ satisfies condition A if it has an asymptotic expansion of the usual type in descending powers of t.

at those of its poles other than $t=0,1,2,\ldots$ which lie to the right of the straight line $\mathscr{R}(\kappa t)=h_2$, and S for the integer satisfying

$$\tfrac{1}{4}\delta\sec\gamma-\tfrac{1}{2}\leqslant S<\tfrac{1}{4}\delta\sec\gamma+\tfrac{1}{2}. \tag{2·13}$$

Theorem 1. *If* (i) $$|\arg X|\leqslant\tfrac{1}{2}\pi,$$

(ii) $$h_1\geqslant\tfrac{3}{2}-\mathscr{R}(\alpha_1) \tag{2·14}$$

and (iii) $\phi(t)$ *is regular and satisfies condition A when* $\mathscr{R}(\kappa t)>h_1$, *then*

$$f(x)=\sum_{s=-S}^{S}I(X_s)+O(X^{h_1+\epsilon}). \tag{2·15}$$

Theorem 2. *If* $\mathscr{R}\left(\dfrac{1}{\kappa}\right)<\tfrac{1}{2}$ *and if conditions* (ii) *and* (iii) *of Theorem* 1 *are satisfied, then*

$$f(x)=\sum_{s=-S}^{S}I(X_s). \tag{2·16}$$

Theorem 3. *If* $\mathscr{R}\left(\dfrac{1}{\kappa}\right)\geqslant\tfrac{1}{2}$ *and if conditions* (i), (ii), *and* (iii) *of Theorem* 1 *are satisfied, then*

$$f(x)=I(X)+O(X^{h_1+\epsilon}).$$

Theorem 4. *If* $\mathscr{R}\left(\dfrac{1}{\kappa}\right)\geqslant\tfrac{1}{2}$, $|\arg X|\leqslant\tfrac{1}{2}\pi-\epsilon'$, *and conditions* (ii) *and* (iii) *of Theorem* 1 *are satisfied, then*

$$f(x)=I(X).$$

Let us assume Theorem 1 for the moment; we shall show that Theorems 2, 3 and 4 are corollaries. First, if

$$\frac{\cos\gamma}{\delta}=\mathscr{R}\left(\frac{1}{\kappa}\right)<\tfrac{1}{2},$$

we have $$|\arg X|\leqslant\frac{\pi\cos\gamma}{\delta}<\tfrac{1}{2}\pi-K.$$

Hence condition (i) of Theorem 1 is satisfied for all x. Also

$$\cos\arg X\geqslant\cos\frac{\pi\cos\gamma}{\delta}>K,\quad \mathscr{R}(X)>K|X|,$$

and so $$X^{h_1+\epsilon}=O(X^{1-\alpha_{M+1}}e^X)=O(X^{\frac{3}{2}-\alpha_{M+1}}e^XE(X))$$

and the term $O(X^{h_1+\epsilon})$ in (2·15) can be absorbed in the error term of $I(X)$. This proves Theorem 2.

If $\mathscr{R}\left(\dfrac{1}{\kappa}\right)\geqslant\tfrac{1}{2}$, i.e. $\delta\sec\gamma\leqslant 2$, then $S=0$. Hence Theorem 3 is merely a special case of Theorem 1. If $|\arg X|\leqslant\tfrac{1}{2}\pi-\epsilon'$, then

$$\mathscr{R}(X)>K|X|,\quad X^{h_1+\epsilon}=O(X^{\frac{3}{2}-\alpha_{M+1}}e^XE(X))$$

and Theorem 4 follows from Theorem 3.

2·2. When $\mathscr{R}\left(\frac{1}{\kappa}\right)<\frac{1}{2}$, Theorem 2 gives the expansion for all x. It only remains to consider the expansion when $\mathscr{R}\left(\frac{1}{\kappa}\right)\geqslant\frac{1}{2}$ for values of x not covered by Theorem 4.

Theorem 5. *If*

(i) $$\mathscr{R}\left(\frac{1}{\kappa}\right)\geqslant\tfrac{1}{2},\quad 0\leqslant\sigma\leqslant\max\left\{1,\pi\left(\frac{\cos\gamma}{\delta}-\frac{1}{2}\right)\right\},\quad |\arg X|\geqslant\tfrac{1}{2}\pi+\sigma,$$

(ii) $\phi(t)$ *is regular when* $\mathscr{R}(\kappa t)>h_2$, *except for a finite number of poles, and*

(iii) $$|\phi(t)|<K|t|^{\mathscr{R}(\beta)+h_2-\frac{3}{2}}e^{\delta\sigma|t|} \tag{2·21}$$

when $\mathscr{R}(\kappa t)>h_2$ *and* $|t|>K$, *then*

$$f(x)=H(-x)+O(X^{h_2+\epsilon}).$$

Theorem 6. *If*

(i) $$\mathscr{R}\left(\frac{1}{\kappa}\right)>\tfrac{1}{2},\quad \sigma>0,\quad |\arg X|\geqslant\tfrac{1}{2}\pi+\sigma,$$

(ii) $\phi(t)$ *is regular when* $\mathscr{R}(\kappa t)>h_2$, *except for a finite number of poles, and*

(iii) $$\phi(t)=O(t^K)$$

when $\mathscr{R}(\kappa t)>h_2$ *and* $|t|>K$, *then*

$$f(x)=H(-x)+O(X^{h_2+\epsilon}).$$

If Theorem 6 is true for any particular $\sigma>0$, it is true for all larger σ. Hence σ may be supposed to satisfy the condition of Theorem 5, and Theorem 6 is an immediate corollary of the latter theorem; for, if $\sigma>0$,

$$|t^K|<K|t|^{\mathscr{R}(\beta)+h_2-\frac{3}{2}}e^{\delta\sigma|t|}.$$

2·3. Theorem 6 gives us the purely algebraic expansion for the relevant values of $\arg X$. We have now only to consider the neighbourhood of the spirals (1·41) and (1·42). Here we have "mixed" expansions. Theorems 3 and 5 give us certain information for these regions, but this is not very precise. For example, if $\arg X=\frac{1}{2}\pi$,

$$I(X)=O(X^{\frac{3}{2}-\alpha_1}e^X)=O(X^{h_1})$$

by (2·14) and so Theorem 3 tells us only that $f(x)=O(X^{h_1+\epsilon})$.

If $\mathscr{R}(1/\kappa)>\frac{1}{2}$, we prove a result uniform in the region $|\arg X|\leqslant\frac{1}{2}\pi+\mu$, where μ is a certain positive number. This region encloses both the spirals (1·42).

Theorem 7. *If*

(i) $$\mathscr{R}\left(\frac{1}{\kappa}\right)>\tfrac{1}{2},\quad 0<\mu<\min\left\{\frac{\pi\cos\gamma}{\delta}-\frac{\pi}{2},\frac{\pi}{2}\right\},$$

(ii) $$|\arg X|\leqslant\tfrac{1}{2}\pi+\mu, \tag{2·31}$$

(iii) $$h_2\geqslant\tfrac{3}{2}-\mathscr{R}(\alpha_{M+1}), \tag{2·32}$$

(iv) *$\phi(t)$ is regular when $\mathscr{R}(\kappa t) > h_2$, except at a finite number of poles, and*

(v) *$\phi(t)$ satisfies condition A when $\mathscr{R}(\kappa t) > h_2$ and $|t| > K$, then*

$$f(x) = I(X) + H(-x) + O(X^{h_2+\epsilon}). \tag{2·33}$$

If $\mathscr{R}(1/\kappa) = \frac{1}{2}$, there is only one spiral, viz.

$$\arg x - \tan\gamma \log|x| = \pi, \tag{2·34}$$

as we saw in § 1·4. On this spiral we have

$$\arg(-x) - \tan\gamma \log|x| = 0.$$

We write $$X' = (-x)^{1/\kappa} e^{\pi i/\kappa}, \quad X'' = (-x)^{1/\kappa} e^{-\pi i/\kappa},$$

where $\arg(-x)$ is determined by (1·32). It is readily verified that

$$X' = X, \; X'' = X_{-1} \,(\arg X > 0); \quad X' = X_1, \; X'' = X \,(\arg X \leqslant 0).$$

We prove a result uniform in the region

$$|\arg(-x) - \tan\gamma \log|x|| \leqslant \pi - \epsilon,$$

i.e. a region enclosing the spiral (2·34).

Theorem 8. *If*

(i) $$\mathscr{R}(1/\kappa) = \tfrac{1}{2}, \quad 0 < \mu' < \tfrac{1}{2}\pi,$$

(ii) $$\tfrac{1}{2}\pi - \mu' \leqslant |\arg X| \leqslant \tfrac{1}{2}\pi, \quad \text{i.e.} \quad |\arg(-x) - \tan\gamma \log|x|| < 2\mu',$$

and conditions (iii), (iv) *and* (v) *of Theorem* 7 *are satisfied, then*

$$f(x) = I(X') + I(X'') + H(-x) + O(X^{h_2+\epsilon}).$$

2·4. Theorems 2, 4, 6, 7 and 8 provide us with the asymptotic expansion of $f(x)$ for all x, provided that $\phi(t)$ satisfies their conditions. The following theorem connects the expansion of $\phi(t)$ postulated in condition A with a more familiar type of expansion.

Theorem 9. *If an integer $N \geqslant 0$ and numbers $a_1, a_2, \ldots, a_N, \mu_1, \mu_2, \ldots, \mu_{N+1}$ are known such that*

$$\mathscr{R}(\mu_1) \leqslant \mathscr{R}(\mu_2) \leqslant \ldots \leqslant \mathscr{R}(\mu_N) < \mathscr{R}(\mu_{N+1})$$

and $$\phi(t) = \sum_{n=1}^{N} a_n t^{-\mu_n} + O(t^{-\mu_{N+1}})$$

for some set of values of t, then an integer $M \geqslant 0$ and numbers $A_1, A_2, \ldots, A_M, \alpha_1, \alpha_2, \ldots, \alpha_{M+1}$ can be calculated such that $\phi(t)$ satisfies condition A for those values of t belonging to the set for which

$$|\arg \kappa t| < \pi, \quad |\arg(\kappa t + \beta)| < \pi, \quad |\arg(\kappa t + \beta + \mu_n)| < \pi \quad (1 \leqslant n \leqslant N+1).$$

In particular $$\alpha_1 = \beta + \mu_1, \quad \alpha_{M+1} = \beta + \mu_{N+1}.$$

This is an immediate consequence of the well-known result

$$\frac{1}{\Gamma(\kappa t + \alpha)} = \left(\frac{e}{\kappa t}\right)^{\kappa t} (\kappa t)^{\frac{1}{2} - \alpha} \left\{ \sum_{l=0}^{L-1} b_l t^{-l} + O(t^{-L}) \right\}, \tag{2·41}$$

in which $b \neq 0$ and b_l is a known function of κ, α and l, and which is valid when

$$|\arg(\kappa t)| < \pi \quad \text{and} \quad |\arg(\kappa t+\alpha)| < \pi.$$

From this we have

$$\frac{t^{-\mu}}{\Gamma(\kappa t+\beta)} = \sum_{l=0}^{L-1} \frac{b'_l}{\Gamma(\kappa t+\beta+\mu+l)} + O\left(\frac{1}{\Gamma(\kappa t+\beta+\mu+L)}\right),$$

and so
$$\frac{\phi(t)}{\Gamma(\kappa t+\beta)} = \sum_{n=1}^{N} \sum_{r=0}^{R_n} \frac{A_{n,r}}{\Gamma(\kappa t+\mu_n+\beta+r)} + O\left(\frac{1}{\Gamma(\kappa t+\mu_{N+1}+\beta)}\right),$$

where
$$R_n + 1 + \mathscr{R}(\mu_n) \geq \mathscr{R}(\mu_{N+1}).$$

This is equivalent to (2·11).

2·5. *Watson's results.* We are now in a position to describe these results in more detail and to explain where we are able to make advances. Watson (1913) proves results equivalent to those of Theorems 2 and 4* with $\phi(t)$ subject to more restricted conditions, viz. (i) that $\phi(t)$ is regular when $\mathscr{R}(\kappa t) > h_1$ and (ii) that there is a positive number λ such that, when $|t| > K$ and $|\arg(\kappa t)| \leq \frac{1}{2}\pi + \lambda$, $\phi(t)$ possesses an asymptotic expansion

$$\phi(t) = t^{-\beta}\left\{\sum_{n=1}^{N} a_n t^{-n} + R_N\right\} \tag{2·51}$$

for *all* N with $|a_n| < C_1 C_2^n n!$ and $|R_N t^{N+1}| < C_3 C_4^N N!$, where C_1, C_2, C_3, C_4 are numbers independent of t, n and N. Condition (ii) enables Watson to use the powerful theory of asymptotic series which he had previously developed (Watson, 1912); in fact, when this condition is satisfied, $\phi(t)$ can be expanded in a *convergent* series of inverse factorials.

Hence in Theorems 2 and 4 my contributions are:

(i) It is enough for $\phi(t)$ to have an asymptotic expansion of the type (2·51) for *some* N and without restriction on the magnitude of the coefficients in the expansion; from this we may find an asymptotic expansion of $f(x)$ to a corresponding degree of approximation.

(ii) The expansion of $\phi(t)$ need not be in inverse *integral* powers.

(iii) The expansion need only hold good for $\mathscr{R}(\kappa t) \geq h_1$ instead of for

$$|\arg(\kappa t)| \leq \tfrac{1}{2}\pi + \lambda.$$

(iv) The results are uniform in $\arg X$ within the limits stated.

Of these (i), (iii) and (iv) are substantial advances, but (ii) is a trivial consequence of (i). In my proof of Theorem 1, I use a transformation due to Watson which greatly simplifies the work.

* Except that "uniformity in $\arg X$" is not proved. For example, Watson's formula (10) on p. 24 of the paper referred to above is not true uniformly in the interval of $\arg y$ given, though it is true for any particular value of $\arg y$ in this interval.

Theorem 6 is implicit in a remark of Watson's. He does not give the proof in detail (it is in any case fairly trivial), and I have felt it worth while to prove the new and more general Theorem 5, which is also useful in the proofs of Theorems 7 and 8.

These last two theorems give the expansion in the important (and troublesome) "barrier regions". So far as I know these theorems are entirely new.

2·6. *A preliminary lemma.* We have proved Theorem 9 and shown that Theorems 2, 3 and 4 follow from Theorem 1, and Theorem 6 from Theorem 5. Hence we have only to prove Theorems 1, 5, 7 and 8. The rest of the paper is devoted to this. Our work is simplified by

Lemma 1. *Without loss of generality we may take* $h_1 \geqslant 0$ *in Theorem* 1 *and* $h_2 \geqslant 0$ *in Theorems* 5, 7 *and* 8.

We prove this assertion for Theorem 7; the proof for the other theorems is similar.

Let us suppose that $h_2 < 0$ and write*

$$j_0 = -\left[\frac{h_2}{\mathscr{R}(\kappa)}\right]$$

so that $j_0 \geqslant 1$. We write†

$$f_0(x) = x^{j_0}\left\{\sum_{n=-j_0}^{-1} \frac{\phi(n)\, x^n}{\Gamma(\kappa n+\beta)} + f(x)\right\}$$

$$= \sum_{n=-j_0}^{\infty} \frac{\phi(n)\, x^{n+j_0}}{\Gamma(\kappa n+\beta)} = \sum_{n=0}^{\infty} \frac{\phi_0(n)\, x^n}{\Gamma(\kappa n+\beta_0)},$$

where $$\phi_0(t) = \phi(t-j_0), \quad \beta_0 = \beta - \kappa j_0.$$

Since $\phi(t)$ satisfies conditions (iv) and (v) of Theorem 7, it follows that $\phi_0(t)$ satisfies the same conditions with

$$h_{2,0} = h_2 + j_0 \mathscr{R}(\kappa) \geqslant 0$$

replacing h_2; in fact,

$$\frac{\phi_0(t)}{\Gamma(\kappa t+\beta_0)} = \frac{\phi(t-j_0)}{\Gamma(\kappa(t-j_0)+\beta)} = \sum_{m=1}^{M} \frac{\kappa A_m}{\Gamma(\kappa(t-j_0)+\alpha_m)} + O\left(\frac{1}{\Gamma(\kappa t+\alpha_{M+1}-\kappa j_0)}\right)$$

$$= \sum_{m=1}^{M} \frac{\kappa A_m}{\Gamma(\kappa t+\alpha_{m,0})} + O\left(\frac{1}{\Gamma(\kappa t+\alpha_{M+1,0})}\right)$$

when $\mathscr{R}(\kappa t) > h_{2,0}$ and $|t| > K$.

* As usual, $[v]$ is the largest integer not greater than v.

† If $\dfrac{\phi(t)}{\Gamma(\kappa t+\beta)}$ has a pole at $t=-l$ $(1 \leqslant l \leqslant j_0)$, we replace $\dfrac{\phi(-l)\, x^{-h}}{\Gamma(\beta-\kappa l)}$ in the first sum by the residue of

$$-k(t) = \frac{\pi\phi(t)\,(-x)^t}{\sin \pi t\; \Gamma(\kappa t+\beta)}$$

at its multiple pole at $t=-l$. The proof of Lemma 1 follows the same lines, while the details of the proof of Theorem 7 for $f_0(x)$ are precisely as before and (2·33) is still true.

Hence, if Theorem 7 is true for $h_2 \geqslant 0$, we have

$$f_0(x) = I_0(X) + H_0(-x) + O(X^{h_{2,0}+\epsilon}),$$

where
$$I_0(X) = Xe^X\left\{\sum_{m=1}^{M} A_m X^{-\alpha_{m,0}} + O(X^{\frac{1}{2}-\alpha_{M+1,0}} E(X))\right\}$$
$$= x^{j_0} I(X),$$

and $H_0(-x)$ is the sum of the residues of

$$k_0(t) = -\frac{\pi\phi_0(t)\,(-x)^t}{\sin \pi t\, \Gamma(\kappa t+\beta_0)} = x^{j_0} k(t-j_0)$$

at its poles to the right of $\mathscr{R}(\kappa t) = h_{2,0}$, except those at $t=1, 2, \ldots$. Now

$$-x^{j_0} \sum_{n=1-j_0}^{-1} \frac{\phi(n)\, x^n}{\Gamma(\kappa n+\beta)} = -\sum_{n=1}^{j=1} \frac{\phi_0(n)\, x^n}{\Gamma(\kappa n+\beta_0)}$$

is the sum of the residues of $k_0(t)$ at its poles at $t=1, 2, \ldots, j_0-1$. Hence

$$L(x) = H_0(-x) - x^{j_0} \sum_{n=1-j_0}^{-1} \frac{\phi(n)\, x^n}{\Gamma(\kappa n+\beta)}$$

is the sum of the residues of $k_0(t)$ at its poles to the right of $\mathscr{R}(\kappa t) = h_{2,0}$ except those at $t = j_0, j_0+1, \ldots$, i.e. the sum of the residues of $x^{j_0}k(t)$ at its poles to the right of $\mathscr{R}(\kappa t) = h_2$ except those at $t=0, 1, 2, \ldots$; and so

$$L(x) = x^{j_0} H(-x).$$

Hence
$$f(x) = x^{-j_0} f_0(x) - \sum_{n=-j_0}^{-1} \frac{\phi(n)\, x^n}{\Gamma(\kappa n+\beta)}$$
$$= I(X) + x^{-j_0} L(x) - \frac{\phi(-j_0)\, x^{-j_0}}{\Gamma(\beta-\kappa j_0)} + O(X^{h_{2,0}-\mathscr{R}(\kappa) j_0+\epsilon})$$
$$= I(X) + H(-x) + O(X^{h_2+\epsilon})$$

by (1·36), and so Theorem 7 for negative h_2 follows from the same theorem for $h_2 \geqslant 0$.

It is clear that the rest of the lemma may be proved similarly. We therefore take $h_1 \geqslant 0$ and $h_2 \geqslant 0$ in the proofs which follow.

Proof of Theorem 1

3·1. *Watson's transformation* (Watson 1913, p. 23). This transformation enables us to deduce Theorem 1 for general κ from results for $f(x)$ when $\kappa = 1$; the latter results, however, have to be found for all values of $\arg X$ and not only for those for which $|\arg X| \leqslant \frac{1}{2}\pi$.

We choose h so that $h_1 < h < h_1 + \epsilon$ and h is neither an integer nor an integral multiple of $\mathscr{R}(\kappa)$. Since we may suppose that $h_1 \geqslant 0$ by Lemma 1, we have $h > 0$. We write

$$\omega = h - h_1 > 0, \quad j = [h], \quad j_1 = \left[\frac{h}{\mathscr{R}(\kappa)}\right], \quad r = |t|, \quad \theta = \arg(\kappa t) \quad (-\pi < \theta \leqslant \pi),$$

$$\psi(t) = \frac{1}{\kappa}\phi\left(\frac{t}{\kappa}\right), \quad F(y) = \sum_{n=j+1}^{\infty} \frac{\psi(n)\, y^n}{\Gamma(n+\beta)},$$

and

$$\chi(t) = \frac{\pi \sin\{(q-\kappa)\, t\pi\}\, \phi(t)\, x^t}{\sin(\kappa t\pi)\sin(t\pi)\,\Gamma(\kappa t+\beta)}.$$

Watson's transformation is contained in

Lemma 2. *If* $q = 2p+1$ *is an odd non-negative integer and*

$$\delta\,|\arg X| \leqslant \tfrac{1}{2}\pi\delta + \pi\cos\gamma - \pi\,|q\cos\gamma - \delta|, \tag{3·11}$$

then

$$f(x) = \sum_{s=-p}^{p} F(X_s) + O(X^h).$$

Watson indicates the proof, but does not give it in full. As I require the lemma under wider conditions than his (e.g. he excludes equality in (3·11)), I give a complete proof.

We denote by D_1 the straight line $\mathscr{R}(\kappa t) = h$ in the t-plane, described in an upward direction. Since

$$j < h < j+1, \quad \mathscr{R}(\kappa)\, j_1 < h < \mathscr{R}(\kappa)\,(j_1+1),$$

the points

$$t = \frac{j+1}{\kappa}, \quad \frac{j+2}{\kappa}, \ldots; \qquad j_1+1, \quad j_1+2, \ldots \tag{3·12}$$

lie to the right of D_1 and the points

$$t = \frac{j}{\kappa}, \quad \frac{j-1}{\kappa}, \ldots; \qquad j_1, \quad j_1 - 1, \ldots$$

to the left of D_1. The points (3·12) are the poles of $\chi(t)$ to the right of D_1.

We observe that $t = re^{i(\theta-\gamma)}$ and that

$$\tfrac{1}{2} - R(\alpha_{M+1}) - h < \tfrac{1}{2} - R(\alpha_1) - h = \tfrac{1}{2} - R(\alpha_1) - h_1 - \omega \leqslant -1 - \omega \tag{3·13}$$

by (2·14).

We require

Lemma 3. *If* (3·11) *is satisfied and if either* (i) $\mathscr{R}(\kappa t) \geqslant h$, *the distance of* t *from each of the points* (3·12) *is greater than* K, *and* $r\,|X|^{-1}$ *is sufficiently large or* (ii) $\mathscr{R}(\kappa t) = h$, *then*

$$|\chi(t)| < Kr^{-1-\omega}\,|X|^h.$$

Let us suppose that one or other set of conditions is fulfilled. Then $\phi(t)$ satisfies condition A and so

$$\left|\frac{\phi(t)}{\Gamma(\kappa t+\beta)}\right| < \frac{K}{|\Gamma(\kappa t+\alpha_1)|}$$

$$< Kr^{\frac{1}{2}-\mathscr{R}(\alpha_1)} \exp\{\delta r(\cos\theta\log(e\delta^{-1}r^{-1}) + \theta\sin\theta)\}$$

by (2·41). Since t is at a finite distance from the poles of $\chi(t)$, we have

$$|\chi(t)| < Kr^{\frac{1}{2}-\mathscr{R}(\alpha_1)}e^{\nu_1},$$

where

$$\nu_1 = \nu_1(r,\theta,x) = r\{\pi\,|q\sin(\theta-\gamma)-\delta\sin\theta| - \pi\delta\,|\sin\theta| - \pi\,|\sin(\theta-\gamma)| + \delta\sin\theta(\theta-\arg X) + \delta\cos\theta\log(e\,|X|\,\delta^{-1}r^{-1})\}$$

$$\leqslant r\{|\sin\theta|(\pi\,|q\cos\gamma-\delta| - \pi\delta - \pi\cos\gamma + \delta\,|\theta| + \delta\,|\arg X|) + \delta r\cos\theta\log(K\,|X|\,r^{-1})\}$$

$$\leqslant \delta r\cos\theta\log(K\,|X|\,r^{-1})$$

by (3·11), since $|\theta| < \frac{1}{2}\pi$.

Now, if condition (ii) is satisfied, $\delta r\cos\theta = h$ so that

$$\nu_1 \leqslant h\log(K\,|X|\,r^{-1}).$$

On the other hand, if condition (i) is satisfied, we may suppose r large enough to make $\log(K\,|X|\,r^{-1}) < 0$, so that

$$\nu_1 \leqslant \delta r\cos\theta\log(K\,|X|\,r^{-1}) \leqslant h\log(K\,|X|\,r^{-1}),$$

since $\delta r\cos\theta \geqslant h$. Hence, in either case,

$$|\chi(t)| \leqslant Kr^{\frac{1}{2}-R(\alpha_1)-h}\,|X|^h \leqslant Kr^{-1-\omega}\,|X|^h$$

by (3·13).

It follows from this lemma that

$$Q = -\frac{1}{2\pi i}\int_{D_1}\chi(t)\,dt$$

converges and that $|Q| \leqslant K\,|X|^h$.

We can now choose a sequence of values of R tending to infinity such that every point on the circle $|t| = R$ is at a distance greater than K from the points (3·12). Consider $\int\chi(t)\,dt$ taken along the arc of this circle to the right of D_1. By Lemma 3 the integral is $O(|X|^h R^{-\omega})$, and so tends to 0 as R tends to infinity. Hence, by Cauchy's theorem, Q is equal to the sum of the residues of $\chi(t)$ at its poles to the right of D_1, that is, at the points (3·12). Hence*

$$Q = \sum_{n=j+1}^{\infty}\frac{\sin(qn\pi/\kappa)\,\phi(n/\kappa)\,X^n}{\kappa\sin(n\pi/\kappa)\,\Gamma(n+\beta)} - \sum_{n=j_1+1}^{\infty}\frac{\phi(n)\,x^n}{\Gamma(\kappa n+\beta)}$$

$$= \sum_{n=j+1}^{\infty}\sum_{s=-p}^{p}\frac{\exp(2sn\pi i/\kappa)\,\phi(n/\kappa)\,X^n}{\kappa\Gamma(n+\beta)} - \sum_{n=j_1+1}^{\infty}\frac{\phi(n)\,x^n}{\Gamma(\kappa n+\beta)}$$

$$= \sum_{s=-p}^{p}\sum_{n=j+1}^{\infty}\frac{\psi(n)\,X_s^n}{\Gamma(n+\beta)} - \sum_{n=j_1+1}^{\infty}\frac{\phi(n)\,x^n}{\Gamma(\kappa n+\beta)}$$

$$= \sum_{s=-p}^{p}F(X_s) - f(x) + O(X^j) + O(x^{j_1}), \tag{3·14}$$

* We appear to assume here that no two of the points (3·12) coincide, i.e. that κ is not a rational number. A little calculation will show that, if two of the points (3·12) coincide, there is still only a single pole at this point and its contribution to our result is such as to leave (3·14) unaltered.

and so we have $$f(x) = \sum_{s=-p}^{p} F(X_s) + O(X^j) + O(x^{j_1}) + O(X^h)$$

$$= \sum_{s=-p}^{p} F(X_s) + O(X^h)$$

by (1·36). This is Lemma 2.

3·2. *The asymptotic expansion of the function* $F(y)$. Let us suppose for the present that $\mathscr{R}(t) \geqslant h > h_1 \geqslant 0$. It follows from the properties of $\phi(t)$ that $\psi(t)$ is regular and that

$$\frac{\psi(t)}{\Gamma(t+\beta)} = \sum_{m=1}^{M} \frac{A_m}{\Gamma(t+\alpha_m)} + O\left(\frac{1}{\Gamma(t+\alpha_{M+1})}\right).$$

Now $$t \geqslant h > h_1 > \tfrac{3}{2} - \mathscr{R}(\alpha_1) \geqslant \tfrac{3}{2} - \mathscr{R}(\alpha_{M+1}) > -\mathscr{R}(\alpha_{m+1}),$$

and so $\Gamma(t+\alpha_{M+1})$ is regular. Hence, if

$$\psi_1(t) = \Gamma(t+\alpha_{M+1}) \left\{ \frac{\psi(t)}{\Gamma(t+\beta)} - \sum_{m=1}^{M} \frac{A_m}{\Gamma(t+\alpha_m)} \right\},$$

$\psi_1(t)$ is regular and $|\psi_1(t)| < K$. Again, if

$$\alpha = \alpha_{M+1}, \quad S(\tau, y) = \sum_{n=0}^{\infty} \frac{y^n}{\Gamma(n+\tau)}, \quad F_1(y) = \sum_{n=j+1}^{\infty} \frac{\psi_1(n)\, y^n}{\Gamma(n+\alpha)},$$

then $$F(y) = \sum_{m=0}^{M} A_m S(\alpha_m, y) + F_1(y) + O(y^j). \tag{3·21}$$

For the moment we assume the following lemmas which we shall prove in §§ 3·3 and 3·4.

Lemma 4. *For all* $y \neq 0$ *and any integer* $P \geqslant 1$

$$S(\tau, y) = y^{1-\tau} e^y - \sum_{p=1}^{P-1} \frac{y^{-p}}{\Gamma(\tau - p)} + O(y^{-P}),$$

where $-\pi < \arg y \leqslant \pi$.

Lemma 5. *Let* $\psi_1(t)$ *be regular and bounded when* $\mathscr{R}(t) \geqslant h$. *If* $\mathscr{R}(y) \leqslant 0$, *then* $F_1(y) = O(y^h)$, *while, if* $R(y) \geqslant 0$, *then*

$$F_1(y) = O(y^h) + O(y^{\frac{3}{2}-\alpha} e^y E(y)).$$

From these two lemmas and (3·21) we deduce at once that

$$F(y) = I(y) + O(y^h) \tag{3·22}$$

for all y.

Let us take $|\arg X| \leqslant \frac{1}{2}\pi$. We choose q in Lemma 2 so that

$$\delta \sec \gamma - 1 \leqslant q < \delta \sec \gamma + 1. \tag{3·23}$$

Then $$\tfrac{1}{2}\pi\delta + \pi \cos \gamma - \pi\, |q \cos \gamma - \delta| \geqslant \tfrac{1}{2}\pi\delta \geqslant \delta\, |\arg X|,$$

and so (3·11) is satisfied. Hence

$$f(x) = \sum_{s=-p}^{p} I(X_s) + O(X^h)$$

by Lemma 2 and (3·22), since $X_s = O(X)$ for $|s| < K$.

To deduce Theorem 1 it only remains to show that

$$\sum_{s=S+1}^{p} I(X_s) + \sum_{s=-p}^{s=-S-1} I(X_s) = O(X^h),$$

since $h < h_1 + \epsilon$. By (2·14), $h > h_1 \geq \frac{3}{2} - \mathscr{R}(\alpha_1)$, and so

$$I(X_s) = O(X_s^{\frac{3}{2}-\alpha_1} e^{X_s}) = O(X^h e^{X_s}).$$

Hence we have only to show that $\mathscr{R}(X_s) \leq 0$ for $S+1 \leq s \leq p$ and for $-p \leq s \leq -S-1$.

By (2·13) and (3·23) $\qquad S \geq \frac{1}{4}\delta \sec\gamma - \frac{1}{2}, \quad p < \frac{1}{2}\delta \sec\gamma.$

Hence for the values of s we are considering

$$|\arg X_s| = \left|\arg X + \frac{2s\pi\cos\gamma}{\delta}\right| \geq \frac{2(S+1)\pi\cos\gamma}{\delta} - |\arg X|$$

$$\geq \frac{\pi\cos\gamma}{\delta}\left(\frac{\delta\sec\gamma}{2}+1\right) - |\arg X|$$

$$= \frac{\pi}{2} + \frac{\pi\cos\gamma}{\delta} - |\arg X| \geq \frac{\pi}{2}$$

by (1·34). Also

$$|\arg X_s| = \left|\arg X + \frac{2s\pi\cos\gamma}{\delta}\right| \leq \frac{2p\pi\cos\gamma}{\delta} + |\arg X| < \pi + |\arg X| \leq \tfrac{3}{2}\pi.$$

Hence $\mathscr{R}(X_s) \leq 0$.

To complete the proof of Theorem 1 we have now to prove Lemmas 4 and 5.

3·3. *Proof of Lemma* 4. We prove a more general result of which Lemma 4 is a particular case.

Lemma 6. *If* $0 < \sigma_1 < \frac{1}{2}\pi$ *and*

$$S(\kappa, \tau; x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\kappa n + \tau)},$$

then for all $x \neq 0$ *and any integer* $P \geq 1$

$$S(\kappa, \tau; x) = \frac{1}{\kappa} \sum_{|\arg X_s| < \frac{1}{2}\pi + \sigma_1} X_s^{1-\tau} e^{X_s} - \sum_{p=1}^{P-1} \frac{x^{-p}}{\Gamma(\tau - p\kappa)} + O(x^{-P}).$$

If we put $\kappa = 1$ and $x = X = y$, this is Lemma 4.

Since Lemma 6 is trivial when $|x| < K$, we may suppose that $|x| > e^{\pi\delta|\sin\gamma|}$. We put $r_1 = 1$ and choose

$$r_2 > |x|^{\delta^{-1}\sec\gamma} e^{\pi|\tan\gamma|}. \tag{3·31}$$

By (1·31) and the definition of X_s,

$$|X_s| = |x|^{\delta^{-1}\sec\gamma} e^{\arg X_s \tan\gamma}$$

and so, if $|\arg X_s| < \pi$, $$r_1 < |X_s| < r_2. \tag{3·32}$$

The contour C_i $(i=1,2)$ consists of three parts, viz.: (i) the real axis from $u=-\infty$ to $u=-r_i$, with $\arg u = -\pi$, (ii) the circle $|u| = r_i$ from $\arg u = -\pi$ to $\arg u = \pi$ and (iii) the real axis from $u=-r_i$ to $u=-\infty$, with $\arg u = \pi$, described in that order. By (3·32) those of the points $u = X_s$ for which $|\arg X_s| < \pi$ lie between C_1 and C_2. If necessary we deform C_1 and C_2 slightly so that $|u^\kappa - x| > K$ on C_1 and so that, if $\arg X_s = \pm\pi$ for any s, the corresponding point $u = X_s$ does not lie between C_1 and C_2.

By (3·31), $|u^\kappa| > |x|$ on C_2 and so, since

$$\frac{1}{\Gamma(\kappa n+\tau)} = \frac{1}{2\pi i}\int_{C_2} u^{-\tau-\kappa n} e^u \, du,$$

we have $$S(\kappa,\tau;x) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{C_2} u^{-\tau} e^u \left(\frac{x}{u^\kappa}\right)^n du = \frac{1}{2\pi i}\int_{C_2}\frac{u^{\kappa-\tau}e^u}{u^\kappa - x}\, du$$

$$= \frac{1}{2\pi i}\int_{C_1}\frac{u^{\kappa-\tau}e^u}{u^\kappa - x}\, du + \frac{1}{\kappa}\sum_{|\arg X_s|<\pi} X_s^{1-\tau} e^{X_s}$$

by Cauchy's theorem, since $u^\kappa = x$ at the points $u = X_s$.

Now we have on C_1, for any $P \geq 1$,

$$\frac{u^\kappa}{u^\kappa - x} = -\sum_{p=1}^{P}\frac{u^{p\kappa}}{x^p} + \frac{u^{(P+1)\kappa}}{x^P(u^\kappa - x)} = -\sum_{p=1}^{P}\frac{u^{p\kappa}}{x^p} + O\left(\frac{u^{(P+1)\kappa}}{x^P}\right).$$

Hence $$\frac{1}{2\pi i}\int_{C_1}\frac{u^{\kappa-\tau}e^u}{u^\kappa - x}\,du = -\sum_{p=1}^{P}\frac{x^{-p}}{2\pi i}\int_{C_1} u^{p\kappa-\tau}e^u\,du + O\left(x^{-P}\int_{C_1}|u^{(P+1)\kappa-\tau}e^u|\,|du|\right)$$

$$= -\sum_{p=1}^{P-1}\frac{x^{-p}}{\Gamma(\tau-p\kappa)} + O(x^{-P}),$$

and so $$S(\kappa,\tau;x) = \frac{1}{\kappa}\sum_{|\arg X_s|<\pi} X_s^{1-\tau}e^{X_s} - \sum_{p=1}^{P-1}\frac{x^{-p}}{\Gamma(\tau-p\kappa)} + O(x^{-P})$$

$$= \frac{1}{\kappa}\sum_{|\arg X_s|<\frac{1}{2}\pi+\sigma_1} X_s^{1-\tau}e^{X_s} - \sum_{p=1}^{P-1}\frac{x^{-p}}{\Gamma(\tau-p\kappa)} + O(x^{-P}),$$

since the rejected terms can be included in the error term.

3·4. *Proof of Lemma* 5. We take $|\arg y| \leq \frac{3}{2}\pi$ and we write $\eta = \arg y$, $t = re^{i\theta}$ and

$$g(t) = \frac{\psi_1(t)\, y^t}{\Gamma(t+\alpha)}.$$

Lemma 7. *If* (i) $\mathscr{R}(t) = h$, *or if* (ii) $\mathscr{R}(t) \geq h$ *and* $r \geq e|y|$, *then*

$$|g(t)| < K|y|^h r^{-1-\omega}\exp\{r\sin\theta(\theta-\eta)\}. \tag{3·41}$$

When $\mathscr{R}(t) \geqslant h$, we have $|\psi_1(t)| < K$ and

$$\frac{1}{\Gamma(t+\alpha)} = \frac{1}{(2\pi)^{\frac{1}{2}}} \frac{e^t}{t^{t+\alpha-\frac{1}{2}}} \left\{1 + O\left(\frac{1}{t}\right)\right\}.$$

Hence $$|g(t)| < Kr^{\frac{1}{2}-\mathscr{R}(\alpha)} \exp\{r\cos\theta \log(e|y|r^{-1}) + r\sin\theta(\theta-\eta)\}. \tag{3·42}$$

If condition (i) is satisfied, then

$$|g(t)| < Kr^{\frac{1}{2}-\mathscr{R}(\alpha)} \exp\{h\log(e|y|r^{-1}) + r\sin\theta(\theta-\eta)\}$$
$$= Kr^{\frac{1}{2}-\mathscr{R}(\alpha)-h} |y|^h \exp\{r\sin\theta(\theta-\eta)\}.$$

The result follows by (3·13). If condition (ii) is satisfied, then

$$\log(e|y|r^{-1}) \leqslant 0, \quad r\cos\theta\log(e|y|r^{-1}) \leqslant h\log(e|y|r^{-1}),$$

and the result follows as before.

Lemma 8. *If* $|\eta| \leqslant \frac{3}{2}\pi$, *then* $\left|F_1(y) - \int_h^\infty g(t)\,dt\right| = O(y^h)$.

Let R be any large number such that $R-\frac{1}{2}$ is an integer and $R > e|y|$, and let θ_0 be the angle defined by $$h = R\cos\theta_0, \quad 0 < \theta_0 < \tfrac{1}{2}\pi.$$

The contours L_1, L_2 each follow the real axis from $t=h$ to $t=+\infty$, but L_1 passes above each point $t=n$ (where n is an integer), while L_2 passes below such points. The contour M_1 is the part of the line $\mathscr{R}(t)=h$ described upwards from the real axis, while the contour M_2 is the part of this line described downwards from the real axis. The contours N_1, N_2 are arcs of the circle $|t|=R$; $0 \leqslant \theta \leqslant \theta_0$ on N_1 and $-\theta_0 \leqslant \theta \leqslant 0$ on N_2. Condition (ii) of Lemma 7 is satisfied on N_1 and N_2. Hence

$$\left|\frac{g(t)}{1-e^{-2\pi it}}\right| < K|y|^h R^{-1-\omega} \exp\{R\sin\theta(\theta-\eta-2\pi)\}$$
$$< K|y|^h R^{-1-\omega}$$

on N_1, since $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ and $\eta \geqslant -\frac{3}{2}\pi$. Similarly

$$\left|\frac{g(t)}{e^{2\pi it}-1}\right| < K|y|^h R^{-1-\omega}$$

on N_2. It follows that $$\int_{N_1} \frac{g(t)\,dt}{1-e^{-2\pi it}}, \quad \int_{N_2} \frac{g(t)\,dt}{e^{2\pi it}-1}$$

each tend to zero as $R \to \infty$. Hence

$$\int_{L_1} \frac{g(t)\,dt}{1-e^{-2\pi it}} = \int_{M_1} \frac{g(t)\,dt}{1-e^{-2\pi it}} = B_1 \quad \text{(say)},$$
$$\int_{L_2} \frac{g(t)\,dt}{e^{2\pi it}-1} = \int_{M_2} \frac{g(t)\,dt}{e^{2\pi it}-1} = B_2 \quad \text{(say)}$$

by Cauchy's theorem. The convergence of these integrals is ensured by Lemma 7.

Using Cauchy's theorem once more, we see that

$$F_1(y) = \int_{L_2} \frac{g(t)\,dt}{e^{2\pi i t}-1} - \int_{L_1} \frac{g(t)\,dt}{e^{2\pi i t}-1}$$

$$= B_2 - B_1 + A_1, \tag{3·43}$$

where

$$A_1 = \int_{L_1} \left\{ \frac{g(t)}{1-e^{-2\pi i t}} - \frac{g(t)}{e^{2\pi i t}-1} \right\} dt$$

$$= \int_{L_1} g(t)\,dt = \int_h^\infty g(t)\,dt, \tag{3·44}$$

since $g(t)$ is regular when $\mathscr{R}(t) \geqslant h$.

On M_1 and M_2, $\mathscr{R}(t) = h$. Hence on M_1 by Lemma 7

$$\left| \frac{g(t)}{1-e^{-2\pi i t}} \right| \leqslant K r^{-1-\omega} \,|\, y \,|^h \exp\{r \sin\theta(\theta-\eta-2\pi)\}$$

$$\leqslant K \,|\, y \,|^h r^{-1-\omega},$$

since $\eta \geqslant -\frac{3}{2}\pi$. It follows that $|\, B_1 \,| < K \,|\, y \,|^h. \quad (3·45)$

Similarly $|\, B_2 \,| < K \,|\, y \,|^h. \quad (3·46)$

Lemma 8 follows at once from (3·43) to (3·46).

For $w > 0$ we write

$$J(w) = \int_{h/w}^\infty v^{\frac{1}{2}-\mathscr{R}(\alpha)} \exp\{w(v-1-v\log v)\}\,dv.$$

Then we shall prove

Lemma 9. *If* $\mathscr{R}(y) \leqslant 0$, *then* $\left| \int_h^\infty g(t)\,dt \right| < K \,|\, y \,|^h. \quad (3·47)$

If $\mathscr{R}(y) > 0$, *then* $\left| \int_h^\infty g(t)\,dt \right| < K \,|\, y \,|^h + K \,|\, y^{\frac{3}{2}-\mathscr{R}(\alpha)} e^y \,|\, J(\mathscr{R}(y)). \quad (3·48)$

We suppose that $0 \leqslant \eta \leqslant \frac{3}{2}\pi$. The proof for negative η is similar.

If $\frac{1}{2}\pi \leqslant \eta \leqslant \frac{3}{2}\pi$, $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ and either of the conditions of Lemma 7 are satisfied, we have

$$|\, g(t) \,| < K \,|\, y \,|^h r^{-1-\omega} \exp\{r \sin\theta(\theta-\eta)\}$$

$$< K r^{-1-\omega} \,|\, y \,|^h.$$

Consider the contour formed of the real axis from $t = h$ to $t = R$, the arc of the circle $|\, t \,| = R$ from $\theta = 0$ to $\theta = \theta_0$ and the straight line $\mathscr{R}(t) = h$ from $\arg t = \theta_0$ to $\arg t = 0$. Within this contour $g(t)$ has no poles and the integral of $g(t)$ along the second and third parts is

$$O\left(y^h \int^R r^{-1-\omega}\,dr\right) = O(y^h)$$

as $R \to \infty$. (3·47) follows by Cauchy's theorem.

If $0 \leqslant \eta < \frac{1}{2}\pi$, $0 \leqslant \theta \leqslant \eta$ and either of the conditions of Lemma 7 are satisfied, we have

$$|g(t)| < Kr^{-1-\omega}|y|^h.$$

Now consider the contour formed of the following four parts: the real axis from $t = h$ to $t = R$; the arc $|t| = R$ from $\arg t = 0$ to $\arg t = \eta$; the straight line $\arg t = \eta$ from $|t| = R$ to $|t| = h \sec \eta$; and the straight line $\mathscr{R}(t) = h$ from $\arg t = \eta$ to $\arg t = 0$. Condition (ii) of Lemma 7 is satisfied on the second part of the contour, while condition (i) is satisfied on the fourth part. Hence $\int g(t)\,dt$ taken along these two parts of the contour is $O(y^h)$ when $R \to \infty$. Hence

$$\int_h^\infty g(t)\,dt = \int_{D_2} g(t)\,dt + O(y^h), \tag{3·49}$$

where D_2 denotes the part of the straight line $\arg t = \eta$ described outwards from the point $t = h + ih\tan\eta$ to infinity. On D_2, by (3·42),

$$|g(t)| < Kr^{\frac{1}{2}-\mathscr{R}(\alpha)} \exp\{r\cos\eta\log(e|y|r^{-1})\}$$
$$= K|y|^{\frac{1}{2}-\mathscr{R}(\alpha)}\, v^{\frac{1}{2}-\mathscr{R}(\alpha)} \exp\{\mathscr{R}(y)\, v(1-\log v)\}$$

if $t = yv$. Hence

$$\left|\int_{D_2} g(t)\,dt\right| < K|y^{\frac{3}{2}-\alpha}| \int_{h/\mathscr{R}(y)}^\infty v^{\frac{1}{2}-\mathscr{R}(\alpha)} \exp\{\mathscr{R}(y)\, v(1-\log v)\}\,dv$$
$$= K|y^{\frac{3}{2}-\alpha}e^y|\, J(\mathscr{R}(y)).$$

(3·48) follows from this and (3·49).

Lemma 5 follows at once from Lemmas 8 and 9 when $\mathscr{R}(y) \leqslant 0$. When $\mathscr{R}(y) > 0$ we have only to prove

Lemma 10. $J(w) < K\min(1, w^{-\frac{1}{2}})$.

3·5. *Proof of Lemma* 10. First we suppose $0 < w \leqslant he^{-1}$, so that $\log(h/w) \geqslant 1$. We write

$$\lambda(v) = wv - w - (wv - h)\log v,$$

so that
$$\lambda'(v) = \frac{h}{v} - w\log v; \quad \lambda''(v) = -\frac{h}{v^2} - \frac{w}{v} < 0.$$

Hence for $v \geqslant h/w$
$$\lambda'(v) \leqslant \lambda'(h/w) = w - w\log(h/w) < 0,$$

and so
$$\lambda(v) \leqslant \lambda(h/w) = h - w.$$

Hence
$$J(w) = \int_{h/w}^\infty v^{\frac{1}{2}-\mathscr{R}(\alpha)-h} \exp\{\lambda(v)\}\,dv$$

$$\leqslant e^{h-w}\int_{h/w}^\infty v^{\frac{1}{2}-\mathscr{R}(\alpha)-h}\,dv < K\int_e^\infty v^{-1-\omega}\,dv = K \tag{3·51}$$

by (3·13).

Next we suppose $w > he^{-1} > K$. If $0 < v < 2$ and if $u = v-1$,

$$\mu(v) = v-1-v\log v = u-(1+u)\log(1+u)$$
$$= -\frac{u^2}{1.2}+\frac{u^3}{2.3}-\dots$$
$$= -\frac{u^2}{2}\left(1-\frac{u}{3}+\frac{u^2}{6}-\dots\right)$$
$$\leqslant -\frac{u^2}{3} = -\frac{(v-1)^2}{3}.$$

We divide the range of integration in $J(w)$ into three parts, in which

$$\frac{h}{w} \leqslant v \leqslant \tfrac{1}{2}, \quad \tfrac{1}{2} \leqslant v \leqslant \tfrac{3}{2}, \quad \tfrac{3}{2} \leqslant v$$

respectively, and call the corresponding integrals J_1, J_2, J_3. Of these parts the first will not exist if $h/w > \frac{1}{2}$, and we then take $J_1 = 0$. If $h/w \leqslant \frac{1}{2}$, that is, if $w > 2h$, we have

$$J_1 = \int_{h/w}^{\frac{1}{2}} v^{\frac{1}{2}-\mathscr{R}(\alpha)} \exp\{w\mu(v)\}\,dv \leqslant \int_{h/w}^{\frac{1}{2}} v^{\frac{1}{2}-\mathscr{R}(\alpha)} \exp\left\{-\frac{w(1-v)^2}{3}\right\} dv$$
$$\leqslant \exp\left(-\frac{w}{12}\right)\int_{h/w}^{\frac{1}{2}} v^{\frac{1}{2}-\mathscr{R}(\alpha)}\,dv$$
$$< K\exp\left(-\frac{w}{12}\right)\max\left(1, w^{\mathscr{R}(\alpha)-\frac{3}{2}+\epsilon}\right) < Kw^{-\frac{1}{2}}.$$

Next $$J_2 \leqslant \int_{\frac{1}{2}}^{\frac{3}{2}} v^{\frac{1}{2}-\mathscr{R}(\alpha)} \exp\{w\mu(v)\}\,dv \leqslant K\int_{\frac{1}{2}}^{\frac{3}{2}} \exp\{-\tfrac{1}{3}w(v-1)^2\}\,dv$$
$$< K\int_{-\infty}^{\infty} \exp(-\tfrac{1}{3}wu^2)\,du = Kw^{-\frac{1}{2}}.$$

Again, when $v > 1$, $\mu'(v) = -\log v < 0$, and so, when $v \geqslant \frac{3}{2}$,

$$\mu(v) \leqslant \mu(\tfrac{3}{2}) = -K, \quad w(\mu(v)-\mu(\tfrac{3}{2})) \leqslant K(\mu(v)-\mu(\tfrac{3}{2})).$$

Hence $$J_3 < \exp\{w\mu(\tfrac{3}{2})\}\int_{\frac{3}{2}}^{\infty} v^{\frac{1}{2}-\mathscr{R}(\alpha)} \exp\{K(\mu(v)-\mu(\tfrac{3}{2}))\}\,dv$$
$$= K\exp(-Kw)\int_{\frac{3}{2}}^{\infty} v^{\frac{1}{2}-\mathscr{R}(\alpha)} \exp\{K\mu(v)\}\,dv < Kw^{-\frac{1}{2}},$$

since the last integral is convergent and independent of w. Since $J(w) = J_1+J_2+J_3$, it follows that

$$J(w) < Kw^{-\frac{1}{2}}, \tag{3·52}$$

when $w > he^{-1}$.

(3·51) and (3·52) together give Lemma 10.

Proof of Theorem 5

4·1. We now suppose the conditions of Theorem 5 to be satisfied. Since

$$|\arg X| \geqslant \tfrac{1}{2}\pi + \sigma,$$

we have $$\delta \sec\gamma(\tfrac{1}{2}\pi+\sigma) \leqslant |\arg x - \tan\gamma \log|x|| \leqslant \pi$$

by (1·31) and (1·33). Hence

$$|\arg(-x) - \tan\gamma \log|x|| \leqslant \pi - \delta\sec\gamma(\tfrac{1}{2}\pi+\sigma)$$

and so $$\pi\cos\gamma + \cos\gamma \arg(-x) - \sin\gamma \log|x| \geqslant \delta(\tfrac{1}{2}\pi+\sigma). \tag{4·11}$$

We choose h so that $h_2 < h < h_2+\epsilon$, h is not an integral multiple of $\mathscr{R}(\kappa)$, and $k(t)$ (defined in (2·12)) has no poles for $h_2 < \mathscr{R}(\kappa t) \leqslant h$. By Lemma 1, we may suppose $h_2 \geqslant 0$, so that $h>0$. We write

$$\omega = h - h_2 > 0, \quad r = |t|, \quad \theta = \arg(\kappa t),$$

and take m a positive integer. We use D_3 to denote the straight line $\mathscr{R}(\kappa t) = h$ and $C_{(m)}$ to denote the arc of the circle $|t| = m+\frac{1}{2}$ to the right of D_3. If $m>K$, no poles of $k(t)$ lie on $C_{(m)}$.

Lemma 11. (i) *On* D_3, $$|k(t)| < Kr^{-1-\omega}|X|^h. \tag{4·12}$$

(ii) $$\lim_{m\to\infty}\int_{C_{(m)}} k(t)\,dt = 0. \tag{4·13}$$

We suppose that t lies on $C_{(m)}$ (with $m>K$) or on D_3 (with $|t|>K$). Then, by (2·21) and (2·41),

$$|k(t)| < Kr^{h_2-1}e^{\nu_2}, \tag{4·14}$$

where $$\nu_2 = \nu_2(r,\theta,x) = r\cos(\theta-\gamma)\log|x| - r\sin(\theta-\gamma)\arg(-x)$$
$$-\pi r|\sin(\theta-\gamma)| + \delta r\cos\theta\{1-\log(\delta r)\} + \delta r\theta\sin\theta + \delta\sigma r.$$

If $\theta>0$ and $\theta>\gamma$, we have

$$\begin{aligned}\nu_2 &= r\cos\theta(\cos\gamma\log|x| + \sin\gamma\arg(-x) + \pi\sin\gamma + \delta\{1-\log(\delta r)\})\\ &\quad + r\sin\theta(\sin\gamma\log|x| - \cos\gamma\arg(-x) - \pi\cos\gamma + \delta\theta) + \delta\sigma r\\ &\leqslant \delta r\cos\theta\log(K|X|r^{-1}) + \delta r\sin\theta(\theta-\tfrac{1}{2}\pi) + \delta\sigma r(1-\sin\theta)\\ &\leqslant \delta r\cos\theta\log(K|X|r^{-1}) + \delta r\{\sin\theta(\theta-\tfrac{1}{2}\pi)+1-\sin\theta\}\end{aligned} \tag{4·15}$$

by (4·11) and since $\sigma \leqslant 1$.

Let us now suppose t on D_3; then $\delta r\cos\theta = \mathscr{R}(\kappa t) = h$. Hence, if $\theta>0$,

$$\theta = \frac{\pi}{2} - \arcsin\left(\frac{h}{r}\right) = \frac{\pi}{2} + O\left(\frac{1}{r}\right),$$

and so $\theta > \gamma$ for $|t|>K$. Hence

$$\nu_2 \leqslant K + h\log(|X|r^{-1})$$

by (4·15) and $$|k(t)| < Kr^{h_2-h-1}|X|^h = Kr^{-1-\omega}|X|^h$$

by (4·14). For $\theta<0$ and $|t|>K$, this result may be proved similarly, while, for $|t|\leqslant K$, we have $|k(t)|<K|X|^h$. This completes the proof of (4·12).

Next, since $$1-(1+w)\cos w = -w+\tfrac{1}{2}w^2+O(w^3)$$

for real w, we can find a number K_1 (of type K) small enough to ensure that

$$1-(1+w)\cos w<0, \tag{4·16}$$

when $0<w\leqslant K_1$. We also take $K_1<\frac{1}{2}\pi-|\gamma|$. We divide $C_{(m)}$ into three parts, viz.

$$C'(\tfrac{1}{2}\pi-K_1\leqslant\theta<\tfrac{1}{2}\pi),\quad C''(|\theta|\leqslant\tfrac{1}{2}\pi-K_1),$$
$$C'''(-\tfrac{1}{2}\pi<\theta\leqslant-\tfrac{1}{2}\pi+K_1).$$

On C', $\theta>0$ and $\theta>\gamma$; hence (4·15) holds good. Writing $w=\frac{1}{2}\pi-\theta$, we have $0<w\leqslant K_1$, and so, by (4·16),

$$\nu_2\leqslant\delta r\cos\theta\log(K|X|r^{-1})+\delta r(1-\cos w-w\cos w)$$
$$\leqslant\delta r\cos\theta\log(K|X|r^{-1})\leqslant h\log(K|X|r^{-1}),$$

if $r=m+\frac{1}{2}$ is large enough to make the logarithm negative. Hence

$$|k(t)|<Kr^{h_2-h-1}|X|^h=Kr^{-1-\omega}|X|^h$$

by (4·14), and so $$\left|\int_{C'}k(t)\,dt\right|<Km^{-\omega}|X|^h\to 0 \tag{4·17}$$
as $m\to\infty$.

We may prove similarly that $$\int_{C'''}k(t)\,dt\to 0 \tag{4·18}$$
as $m\to\infty$.

Finally, on C'', we have $\cos\theta\geqslant\sin K_1$. It follows from the definition of ν_2 that

$$\nu_2\leqslant Kr(1+\log|X|)-\delta r\sin K_1\log r$$

and so $$\int_{C''}k(t)\,dt\to 0 \tag{4·19}$$

as $m\to\infty$. (4·13) is an immediate consequence of (4·17), (4·18) and (4·19).

Now let $j_1=[h/\mathscr{R}(\kappa)]$. It follows from (4·13) by Cauchy's theorem that

$$f(x)-\sum_{n=0}^{j_1}\frac{\phi(n)\,x^n}{\Gamma(\kappa n+\beta)}=\frac{1}{2\pi i}\int_{D_3}k(t)\,dt+H(-x)$$

and so $$|f(x)-H(-x)|\leqslant O(x^{j_1})+\frac{1}{2\pi}\left|\int_{D_3}k(t)\,dt\right|$$
$$=O(X^h)+O\left(X^h\int^{\infty}r^{-1-\omega}\,dr\right)$$
$$=O(X^h)=O(X^{h_2+\epsilon}).$$

This is Theorem 5.

Proof of Theorems 7 and 8

5·1. Let us suppose that conditions (iii), (iv) and (v) of Theorem 7 are satisfied. We write $\alpha = \alpha_{M+1}$,

$$\phi_1(t) = \Gamma(\kappa t+\alpha)\left\{\frac{\phi(t)}{\Gamma(\kappa t+\beta)} - \sum_{m=1}^{M} \frac{\kappa A_m}{\Gamma(\kappa t+\alpha_m)}\right\}$$

and
$$f_1(x) = \sum_{n=0}^{\infty} \frac{\phi_1(n)\, x^n}{\Gamma(\kappa n+\alpha)}.$$

Then, when $\mathscr{R}(\kappa t) > h_2$, $\phi_1(t)$ is regular apart from a finite number of poles; also

$$|\,\phi_1(t)\,| < K \tag{5·11}$$

provided $\mathscr{R}(\kappa t) > h_2$ and $|\,t\,| > K$. We have also

$$f(x) = \sum_{m=1}^{M} \kappa A_m S(\kappa, \alpha_m;\, x) + f_1(x)$$

in the notation of Lemma 6 and so, by that lemma,

$$\begin{aligned} f(x) &= \sum_{|\arg X_s| < \frac{1}{2}\pi+\sigma_1} X_s e^{X_s}\left\{\sum_{m=1}^{M} A_m X_s^{-\alpha_m}\right\} + f_1(x) + O(x^{-1}) \\ &= \sum_{|\arg X_s| < \frac{1}{2}\pi+\sigma_1} I(X_s) + f_1(x) + O(x^{-1}). \end{aligned} \tag{5·12}$$

So far we have made no assumptions with regard to κ and $\arg X$. Now let us take

$$\mathscr{R}\left(\frac{1}{\kappa}\right) > \tfrac{1}{2}, \quad |\arg X| \leqslant \tfrac{1}{2}\pi + \mu, \tag{5·13}$$

where μ satisfies condition (i) of Theorem 7. We take $\sigma_1 = \mu$ in (5·12); this is legitimate since $0 < \mu < \frac{1}{2}\pi$. Now, if $s \neq 0$,

$$\begin{aligned} |\arg X_s| = \left|\arg X + \frac{2s\pi\cos\gamma}{\delta}\right| &\geqslant \frac{2\pi\cos\gamma}{\delta} - |\arg X| \\ &\geqslant \frac{2\pi\cos\gamma}{\delta} - \frac{\pi}{2} - \sigma_1 > \frac{\pi}{2} + \sigma_1, \end{aligned}$$

since
$$\sigma_1 = \mu < \frac{\pi\cos\gamma}{\delta} - \frac{\pi}{2}.$$

Hence $|\arg X_s| < \frac{1}{2}\pi + \sigma_1$ is true only for $s = 0$ and the first sum on the right-hand side of (5·12) contains one term only, viz. $I(X)$. Hence

$$f(x) = I(X) + f_1(x) + O(x^{-1}), \tag{5·14}$$

whenever (5·13) is satisfied.

Next let us take
$$\mathscr{R}\left(\frac{1}{\kappa}\right) = \tfrac{1}{2}, \quad \tfrac{1}{2}\pi - \mu' \leqslant \arg X \leqslant \tfrac{1}{2}\pi, \tag{5·15}$$

where μ' satisfies condition (i) of Theorem 8. Then $X' = X$ and $X'' = X_{-1}$. Also

$$\arg X_s = \arg X + s\pi,$$

and so, if $s \neq 0$ and $s \neq -1$, $|\arg X_s| \geqslant \pi > \frac{1}{2}\pi + \mu'$.

On the other hand, $|\arg X_{-1}| \leqslant \frac{1}{2}\pi + \mu$. Hence, if we take $\sigma_1 = \mu' < \frac{1}{2}\pi$ in (5·12), that formula becomes

$$f(x) = I(X') + I(X'') + f_1(x) + O(x^{-1}). \tag{5·16}$$

Similarly we may prove (5·16) when

$$\mathscr{R}\left(\frac{1}{\kappa}\right) = \tfrac{1}{2}, \quad -\tfrac{1}{2}\pi \leqslant \arg X \leqslant -\tfrac{1}{2}\pi + \mu'.$$

Since we may suppose $h_2 \geqslant 0$ by Lemma 1, Theorems 7 and 8 follow at once from (5·14), (5·16) and the following lemma.

Lemma 12. *If* $\mathscr{R}\left(\frac{1}{\kappa}\right) \geqslant \frac{1}{2}, \quad h_2 \geqslant 0, \quad |\arg X| \leqslant \frac{1}{2}\pi + \mu,$

where $0 < \mu < \min\left(\frac{\pi \cos \gamma}{\delta} - \frac{\pi}{2}, \frac{\pi}{2}\right)$

when $\mathscr{R}\left(\frac{1}{\kappa}\right) > \frac{1}{2}$ *and* $\mu = 0$ *when* $R\left(\frac{1}{\kappa}\right) = \frac{1}{2}$, *and if conditions* (iii), (iv) *and* (v) *of Theorem* 7 *are satisfied, then*

$$f_1(x) = H(-x) + O(X^{h_2+\epsilon}) + O(X^{\frac{3}{2}-\alpha} e^X E(X)). \tag{5·17}$$

We write $H_1(-x)$ for the sum of the residues of

$$k_1(t) = -\frac{\pi \phi_1(t)\,(-x)^t}{\sin \pi t\, \Gamma(\kappa t + \alpha)}$$

at its poles to the right of the line $\mathscr{R}(\kappa t) = h_2$ other than those at $t = 1, 2, \ldots$. Then $H_1(-x) - H(-x)$ is the sum of the residues of

$$k_1(t) - k(t) = \frac{\pi(-x)^t}{\sin \pi t}\left(\frac{\phi(t)}{\Gamma(\kappa t + \beta)} - \frac{\phi_1(t)}{\Gamma(\kappa t + \alpha)}\right)$$

$$= \frac{\pi(-x)^t}{\sin \pi t} \sum_{m=1}^{M} \frac{\kappa A_m}{\Gamma(\kappa t + \alpha_m)}$$

at its poles to the right of the line $\mathscr{R}(\kappa t) = h_2$ except those at $t = 1, 2, \ldots$. But $h_2 \geqslant 0$ and so $k_1(t) - k(t)$ has no other poles to the right of $\mathscr{R}(\kappa t) = h_2$. Hence

$$H_1(-x) = H(-x). \tag{5·18}$$

5·2. *Proof of* (5·17) *when* $|\arg X| \geqslant \frac{1}{2}\pi$. In Theorem 5 we write 0 for σ, α for β, $\phi_1(t)$ for $\phi(t)$, $f_1(x)$ for $f(x)$ and $H_1(-x)$ for $H(-x)$. Conditions (i) and (ii) of Theorem 5 are satisfied and condition (iii) becomes

$$|\phi_1(t)| < K|t|^{\mathscr{R}(\alpha)+h_2-\frac{3}{2}}. \tag{5·21}$$

Now
$$\mathscr{R}(\alpha)+h_2-\tfrac{3}{2} \geqslant 0$$

by (2·32), and so (5·21) is a consequence of (5·11). Hence Theorem 5 gives the result

$$f_1(x) = H_1(-x)+O(X^{h_2+\epsilon}) = H(-x)+O(X^{h_2+\epsilon})$$

by (5·18); and (5·17) follows.

5·3. *Proof of* (5·17) *when* $|\arg X| < \frac{1}{2}\pi$.* We write $\xi = \arg X$ and choose K_2 so that

$$0 < K_2 < \tfrac{1}{2}\pi - \max_{t'} |\arg(\kappa t')|,$$

where t' runs through all the poles of $k_1(t)$ to the right of $\mathscr{R}(\kappa t) = h_2$.

First let $|\xi| \leqslant \frac{1}{2}\pi - K_2$. We can choose h_1 so that $\phi_1(t)$ is regular and satisfies (5·11) wherever $\mathscr{R}(\kappa t) > h_1$. Then the conditions of Theorem 4 are satisfied with $M = 0$, $\epsilon' = K_2$, $f_1(x)$ for $f(x)$, $\phi_1(t)$ for $\phi(t)$ and α for β; and so $f_1(x) = O(X^{1-\alpha}e^X)$. Since $\mathscr{R}(X) \geqslant |X| \sin K_2 > K|X|$, we have

$$H(-x) = O(X^K) = O(X^{1-\alpha}e^X)$$

and (5·17) follows at once.

Next let $\frac{1}{2}\pi - K_2 \leqslant |\xi_2| < \frac{1}{2}\pi$. We shall suppose that

$$\tfrac{1}{2}\pi - K_2 \leqslant \xi < \tfrac{1}{2}\pi; \tag{5·31}$$

the proof for $-\frac{1}{2}\pi < \xi < -\frac{1}{2}\pi + K_2$ is exactly similar.

From (5·31) it follows that $0 < \xi < \frac{1}{2}\pi$ and so that

$$0 < \arg x - \tan\gamma \log|x| < \tfrac{1}{2}\pi\delta \sec\gamma \leqslant \pi.$$

Hence
$$-\pi < \arg(-x) - \tan\gamma \log|x| < 0,$$

and so
$$\arg(-x) = \arg x - \pi.$$

Hence
$$k_1(t) = -\frac{\pi\phi_1(t)\,(-x)^t}{\sin \pi t\,\Gamma(\kappa t+\alpha)} = -\frac{2\pi i \phi_1(t)\,x^t}{(e^{2\pi i t}-1)\,\Gamma(\kappa t+\alpha)}.$$

We choose h so that $h_2 < h < h_2+\epsilon$, h is not an integral multiple of $\mathscr{R}(\kappa)$, and $k_1(t)$ has no poles for which $h_2 < \mathscr{R}(\kappa t) \leqslant h$. Since $h_2 \geqslant 0$, we have $h > 0$. We write

$$j_1 = [h/\mathscr{R}(\kappa)], \quad \omega = h - h_2 > 0, \quad r = |t|, \quad \theta = \arg(\kappa t).$$

* The reader will note that the subsequent proof and result is uniform in $\arg X$ in this interval; that is, the constants implied in the error terms do not depend on $\arg X$ however near $\arg X$ may be to $\pm\frac{1}{2}\pi$. A trivial modification would make the proof apply to $\arg X = \pm\frac{1}{2}\pi$, but this is unnecessary in view of what we have proved in §5·2.

The number m is a positive integer, $R = m+\frac{1}{2}$, and the contour Γ_m is made up of the three parts:

$$\Gamma'_m\left(r = R, -\frac{\pi}{2}+\arc\sin\left(\frac{h}{\delta R}\right) \leqslant \theta \leqslant \xi\right),$$

$$\Gamma''_m\,(\theta = \xi,\ h\delta^{-1}\sec\xi \leqslant r \leqslant R),$$

$$\Gamma'''_m\left(\mathscr{R}(\kappa t) = h, -\frac{\pi}{2}+\arc\sin\left(\frac{h}{\delta R}\right) \leqslant \theta \leqslant \xi\right),$$

the whole described in a counter-clockwise direction. We write

$$G_m = \frac{1}{2\pi i}\int_{\Gamma_m} k_1(t)\,dt = G'_m + G''_m + G'''_m,$$

where G'_m, G''_m, G'''_m are the corresponding integrals along Γ'_m, Γ''_m, Γ'''_m. The results of letting $m\to\infty$ in Γ''_m and Γ'''_m are denoted by Γ'' and Γ''', while the limits of G_m, G'_m, G''_m, G'''_m as $m\to\infty$ are denoted (if they exist) by G, G', G'', G'''.

By our choice of K_2 there is no pole t' of $k_1(t)$ such that $\mathscr{R}(\kappa t') \geqslant h$ and $\arg(\kappa t') \geqslant \frac{1}{2}\pi - K_2$. Since $\xi > \frac{1}{2}\pi - K_2$, it follows that all the poles of $k_1(t)$ to the right of the line $\mathscr{R}(\kappa t) = h$ lie to the right of the contour formed by Γ'' and Γ'''. Hence, by Cauchy's theorem, G (if it exists) is equal to the sum of the residues of $k_1(t)$ to the right of the straight line $\mathscr{R}(\kappa t) = h$, i.e.

$$G = H_1(-x) - \sum_{n=j_1+1}^{\infty} \frac{\phi_1(n)\,x^n}{\Gamma(\kappa n+\alpha)}$$

$$= H_1(-x) - f_1(x) + O(x^{j_1}).$$

Hence, by (1·36) and (5·18),

$$f_1(x) = H(-x) + G + O(X^h). \tag{5·32}$$

We take m large enough to ensure that every point of Γ'''_m, and so every point of Γ_m, is a finite distance from every pole of $k_1(t)$. Let t lie on Γ_m. Then

$$|\,e^{2\pi i t}-1\,| > K\max\{1, e^{-2\pi r\sin(\theta-\gamma)}\}$$

and, by (2·41) and (5·11), $\qquad |\,k_1(t)\,| < K r^{\frac{1}{2}-\mathscr{R}(\alpha)} e^{\nu_3},$

where $\qquad \nu_3 = r\{\cos(\theta-\gamma)\log|\,x\,| - \sin(\theta-\gamma)\arg x + \delta\cos\theta(1-\log(\delta r))$

$$+\delta\theta\sin\theta + \min(0, 2\pi\sin(\theta-\gamma))\}$$

$$= \delta r\cos\theta\log\left(\frac{e\,|\,X\,|}{\delta r}\right) + \delta r\sin\theta(\theta-\xi) + \min\{0, 2\pi r\sin(\theta-\gamma)\}.$$

Now $-\frac{1}{2}\pi < \theta \leqslant \xi$. If $0 \leqslant \theta \leqslant \xi$,

$$\nu_3 \leqslant \delta r\cos\theta\log(e\,|\,X\,|\,\delta^{-1}r^{-1}) \tag{5·33}$$

$$= \delta r\cos\theta\log(K\,|\,X\,|\,r^{-1}).$$

If $-\frac{1}{2}\pi<\theta\leqslant 0$, then $\sin\theta\leqslant 0$ and

$$\theta-\xi+\frac{2\pi}{\delta}\cos\gamma\geqslant\frac{2\pi}{\delta}\cos\gamma-\pi\geqslant 0,$$

since
$$\frac{\cos\gamma}{\delta}=\mathscr{R}\left(\frac{1}{\kappa}\right)\geqslant\tfrac{1}{2}$$

and so
$$\nu_3\leqslant\delta r\cos\theta\left\{\log\left(\frac{e\,|\,X\,|}{\delta r}\right)-\frac{2\pi}{\delta}\sin\gamma\right\}+\delta r\sin\theta\left(\theta-\xi+\frac{2\pi\cos\gamma}{\delta}\right)$$

$$\leqslant\delta r\cos\theta\left\{\log\left(\frac{e\,|\,X\,|}{\delta r}\right)-\frac{2\pi}{\delta}\sin\gamma\right\}\leqslant\delta r\cos\theta\log\left(\frac{K\,|\,X\,|}{r}\right).$$

Hence on Γ_m
$$\nu_3\leqslant\delta r\cos\theta\,\log\,(K\,|\,X\,|\,r^{-1}). \tag{5·34}$$

We now consider Γ'_m. On this contour

$$\delta r\cos\theta\geqslant h,\quad \log\,(K\,|\,X\,|\,r^{-1})<0,$$

provided R is sufficiently large. Hence

$$\nu_3\leqslant h\log\,(K\,|\,X\,|\,R^{-1})$$

by (5·34) and so
$$|\,k_1(t)\,|<KR^{\frac{1}{2}-\mathscr{R}(\alpha)-h}\,|\,X\,|^h\leqslant K\,|\,X\,|^h\,R^{-1-\omega}$$

by (2·32). Hence
$$|\,G'_m\,|<K\,|\,X\,|^h\,R^{-\omega}\to 0$$

as $R\to\infty$, i.e. $G'=0$.

Next we consider Γ'''_m. On this $\delta r\cos\theta=h$ and so, by (5·34),

$$\nu_3\leqslant h\log\,(K\,|\,X\,|\,r^{-1}),\qquad |\,k_1(t)\,|<K\,|\,X\,|^h\,r^{-1-\omega}.$$

Hence G''' exists and $G'''=O(X^h)$.

Finally we consider Γ''_m. By (5·33),

$$\nu_3\leqslant\delta r\cos\xi\,\log\,(e\,|\,X\,|\,\delta^{-1}r^{-1})$$

and so G'' exists and

$$|\,G''\,|\leqslant K\int_{h\delta^{-1}\sec\xi}^{\infty}r^{\frac{1}{2}-\mathscr{R}(\alpha)}\exp\left\{\delta r\cos\xi\,\log\left(\frac{e\,|\,X\,|}{\delta r}\right)\right\}dr$$

$$=K\,|\,X\,|^{\frac{3}{2}-\mathscr{R}(\alpha)}\int_{h/\mathscr{R}(X)}^{\infty}v^{\frac{1}{2}-\mathscr{R}(\alpha)}\exp\{\mathscr{R}(X)\,v(1-\log v)\}\,dv$$

$$=K\,|\,X^{\frac{3}{2}-\mathscr{R}(\alpha)}e^X\,|\,J(\mathscr{R}(X))=O(X^{\frac{3}{2}-\mathscr{R}(\alpha)}e^X\,E(X))$$

by Lemma 10.

Hence G exists and

$$G=G'+G''+G'''=O(X^h)+O(X^{\frac{3}{2}-\mathscr{R}(\alpha)}\,e^X\,E(X)). \tag{5·35}$$

Finally (5·17) is a consequence of (5·32) and (5·35), since $h<h_2+\epsilon$. This completes the proof of Lemma 12.

Summary

The function considered is

$$f(x) = c_0 + c_1 x + c_2 x^2 + \dots,$$

where
$$c_n = \frac{\phi(n)}{\Gamma(\kappa n + \beta)},$$

κ and β may be real or complex, $\mathscr{R}(\kappa) > 0$ and $\phi(t)$ is regular and has an asymptotic expansion in descending powers of t not necessarily integral powers to the right of the line $\mathscr{R}(\kappa t) = h_1$.

If $\mathscr{R}\left(\frac{1}{\kappa}\right) < \frac{1}{2}$, the function has one or more exponentially large expansions for all large x in the complex x-plane. If $\mathscr{R}\left(\frac{1}{\kappa}\right) > \frac{1}{2}$, the plane is divided by two spirals into two connected parts; in the interior of one part an exponentially large expansion is valid, in the interior of the other the expansion is algebraic, while in the neighbourhood of the spirals the expansion is mixed, i.e. a sum of the expansions in the two regions. If $\mathscr{R}\left(\frac{1}{\kappa}\right) = \frac{1}{2}$ the spirals coincide; there is an exponentially large expansion at a distance from the spiral and a mixed expansion in the neighbourhood of the spiral.

The results for the single expansion are similar to Watson's, but our conditions on $\phi(t)$ are less severe than his. The results for the mixed expansions are new. Amongst other applications, the latter would enable us to determine the distribution of the zeroes of the function very precisely.

Particular examples of the function have been studied by Mittag-Leffler, Barnes, Hardy and others. The results here include theirs as special cases.

The calculation of the coefficients in the exponential expansion is greatly shortened and the results are given in a simple form which should facilitate applications to particular problems. The method of proof is based on Cauchy's theorem and on principles similar to those of the method of steepest descent, but the complications of the latter method are wholly avoided.

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